Cosmological models

- *Can a galaxy move faster than light?*
- *How old is the Universe and how big is the visible Universe?*
- *When did the dark energy conquer the matter?*

Abstract
In this article we begin by presenting the coordinate system (distance, time) most often used in general relativity. We continue by introducing the scale factor and the Hubble flow and we show how the Hubble law can be deduced. We show how the radial velocity of an object can be expressed as a sum of two parts: a general expansion velocity and a local velocity – the local part giving a deviation from the Hubble expansion. We continue by examining the movement of light in the expanding Universe. And we shall see that a galaxy cannot move faster than light.
We use the Milne Model of the Universe to shed some light on the movement of galaxies. In this model (the empty Universe) we can globally use two coordinate systems: a generally relativistic and a special relativistic coordinate system. It is illustrative to compare the description of the movements of galaxies using these two very different coordinate systems. In the Milne model we can directly translate the velocity in general relativistic coordinates to special relativistic coordinates and show that no 'galaxy' is moving faster than light.
We look at the forces that govern the evolution of the scale factor (matter and dark energy) and mentions three exact solutions of the equation of motion, among these the pt. preferred model of the Universe. We use these models to calculate the age of the Universe(s) using the cosmological parameters as measured by the Planck Satellite.
Also we calculate the size of the visible Universe using the pt. preferred model again using Planck Satellite data.
We argue for the slowdown in pace of events as seen on cosmological distances. The 'video' received from great distances is seen in slow motion.
Finally we argue that viewing the cosmological redshift as a continuous Doppler shift in the expanding Universe is a natural consequence of the equivalence principle and the cosmological principle.

The Coordinate System

In cosmology it is common to use a coordinate system based in the position of the observer being at rest relative to the Hubble flow (in reality: at rest relative to the cosmological background radiation – it means that the observer sees the same radiation temperature in all directions without any dipole contribution). The radial distance from the observer to an object at the time t is denoted by $s(t)$. But there are many different distance concepts in the expanding Universe. As there are different time concepts. What is the more precise definition of distance and time that is most common in use?

Distances

We shall use the *proper distance* $s(t)$ between the observer and the object of interest. That means (on a given time t – see below) the sum of distances as measured between galaxies close to each other in the direction of the object of interest – the galaxies all following the Hubble flow (see below). The distance between two galaxies close to each other (with small relative velocity) should be measured in their rest-system (proper distance).

No matter how fast the galaxies moves as seen by the observer there is no (Lorentz-) contraction of distance in this definition – as opposed to distances in special relativity.
Movements orthogonal to the radial distance will involve an angular part that also describes the eventual curvature of space. This is relevant in distance measures like angular diameter distance or luminosity distance. We will not define these distance measures in this text. They all can be expressed as a function of the proper distance and the scale factor $R(t)$.

Time

We will use the proper time t from the Big Bang and ahead for a cosmological observer at rest relative to the Hubble flow. Expressed otherwise: the time as measured by a watch at rest relative to a galaxy following the expansion of space without peculiar motion. It is denoted as the cosmological (proper-) time.

If we denote the present radial distance to the object of interest s_0, we can express the time evolution of the distance/position of the object of interest by the equation

(1) $s(t) = R(t) \cdot s_0$ Time evolution of distance to object following the Hubble flow

where $R(t)$ is a common scale-factor for all objects following the expansion of space.

If you think of the balloon model of the Universe the scale factor is a kind of 'blow-up' factor.

For all objects following the expansion without peculiar motion the value s_0 will be a constant. For these moving objects the scale factor contains the whole time dependence of the radial distance.

The distance s_0 is often denoted as a co-moving distance and is a constant of time for objects following the expansion of space. This however does not mean that the object do not move relative to the observer – it only means that the time dependence of the position is given by eq. (1).

Given equation (1) we can argue that the value of the scale factor now (time t_0) is 1, that is $R(t_0) = 1$.

The time dependence of the scale factor $R(t)$ is governed by the cosmological differential equation, see below. Thus it is the cosmological gravitational forces that governs the time evolution of the proper distance $s(t)$.

We will now pursue other consequences of equation (1). We begin looking at the Hubble law.

We now looks at a galaxy/observer that is following the Hubble flow. The comoving distance s_0 from us (at the center of the coordinate system) to the galaxy/remote observer is then a constant. Therefore the radial velocity of the galaxy/remote observer is given by

(2) $v(t) = s'(t) = R'(t) \cdot s_0$

This equation shows that the velocity at any given time t is proportional to the present proper distance s_0.

We will transform this equation as follows.

We begin by defining the Hubble-parameter $H(t)$ at a given time t:

(3) $H(t) = \frac{R'(t)}{R(t)}$ Hubble parameter at the time t

The Hubble parameter therefore is the expansion rate of the Universe at a given time.

Using this definition we transform equation (2) as follows:

$$v(t) = s'(t) = R'(t) \cdot s_0 = \frac{R'(t)}{R(t)} \cdot R(t) \cdot s_0 = H(t) \cdot s(t)$$

where we have been using equation (1).

Thus we have shown that as a consequence of equation (1) the Hubble law is valid at all times:

(4) $v(t) = H(t) \cdot s(t)$ Hubble law, time t

For all galaxies following the expansion of the Universe at any given time t the radial velocity is proportional to the proper distance at the same time.

At the present time (time t_0) we have

(5) $v(t_0) = H(t_0) \cdot s_0$ Hubble law, time t_0 (now)

The present value of the Hubble parameter H_0 is $0.0692/\text{Gyr}$ or 6.92% per Gyr ($1\ \text{Gyr} = 10^9\ \text{yr}$).

It should be noted that the velocity of a galaxy $v(t)$ (or $v(t_0)$) is a sum of local velocity differences between a series of galaxies/remote observers following the Hubble flow located at the track of the light from us to the remote galaxy. This is a consequence of the definition of the proper distance $s(t)$ as described above. This is the reason why there is no upper limit on the velocity of the galaxies – as it is the sum of local velocity differences. The only limit to the velocity would be in a Universe limited in distance (a Universe of positive curvature – see below).

The radial movement of light in the Universe

Almost all of the information we receive from cosmological distances is in the form of EM-radiation ('light'). It is therefore important how the light moves in the expanding Universe.

If an object (light, galaxy..) does not follow the Hubble flow and has its own peculiar movement, the radial velocity of the object can be expressed as a sum of two contributions: an expansion part given by equation (4) and a local velocity $v_{\text{local}}(t)$ relative to the Hubbleflow. The combined velocity is (as it follows from equation (1))

$$v_{radial}(t) = s'(t) = H(t) \cdot s(t) + v_{\text{local}}(t)$$

In the case of light we have $v_{\text{local}} = \pm c$. The radial velocity of light therefore is

(6) $v_{radial}(t) = s'(t) = H(t) \cdot s(t) \pm c$ Radial velocity of light at time t

The constant *c* is approximately 300 000 km/s.

If we denote the expansion velocity at the place of the light by $v_{\text{Hubble}} = H(t) \cdot s(t)$, the formula for the velocity of light becomes

$$v_{radial,light} = v_{\text{Hubble}} \pm c$$

Now we are able to answer the question:

Can a galaxy move faster than light?

In an infinite Universe it is natural to pose the question: can a galaxy move faster than light? When we inspect the Hubble law (equation (4)) the velocity of the galaxy is proportional to the proper distance. And if there are no limit on this distance (in a critical or subcritical Universe, see below) there also will be no limits on the velocity. Does that mean that the galaxy moves faster than light?

The answer is no. In this case the so-called Milne Model of the Universe is especially illuminating, see next chapter.

Here we take a look at an example. Following equation (6) we get

$$v_{radial,light} = v_{\text{Hubble}} \pm c$$

The expansion velocity $v_{\text{Hubble}}(t) = H(t) \cdot s(t)$ at the place of the light/photon could be $10c$ for a suitably value of the distance s – and the velocity of light then will be either $10c + c = 11c$ for light moving away from the galaxy in the direction away to us or $10c - c = 9c$ if the light moves in our direction as seen from the galaxy.

That is - the velocity of the galaxy is well within the limits of the speed of light! No superluminal galaxy.

As we noted above the reason for velocities exceeding the value c is the choice of radial distance measure (the proper distance).

Now to the question: will light emitted in our direction (as seen from the emitting galaxy) always be able to reach us? The answer depends on which cosmological model we look at. In some models the answer is yes – for example the model we think is the best current model.

If the expansion velocity at the place of the light/photon never gets below c we will never see the light – as the light/photon is moving away from us at all times. The proper distance is increasing and never becomes 0 (where we are!). We talk here about horizons of two types (past- and future-horizons). In this sense there are parts of space-time we will never be able to communicate with. And galaxies that we will never see.

Milne-cosmology – very abbreviated

There is one cosmological model where we can choose two kinds of coordinate systems. The one coordinate system is as described above (time and radial distance). This we will call the GR-system (GR=general relativity). The other possible coordinate system is defined as in special relativity. We will denote this coordinate system SR (SR=special relativity). The reason that we can use SR is that the Milne Universe is empty! No gravitational forces acts on the 'particles' in this Universe. Therefore the velocity of a 'galaxy' is constant (no acceleration). Nevertheless this model is illustrative because you can translate directly from GR-velocities to SR-velocities.

The translation from the galaxy GR-velocity v_{GR} to the corresponding galaxy SR-velocity v_{SR} is

(7) $$v_{SR} = c \cdot \tanh(v_{GR}/c)$$

where $v_{GR} = H_0 \cdot s_0$ is radial velocity of the 'galaxy' in the infinite Universe described by the proper distance s_0 and galaxy proper time and the corresponding distances and times as described in SR. Take a look at figure 1 below.

The velocity of light in SR-coordinates is of course $\pm c$ everywhere in the coordinate system.

Equation (7) shows that the SR-velocity of the 'galaxy' v_{SR} is approaching the speed of light c as v_{GR}/c is approaching infinity. This follows from the fact that $\tanh(x) = (e^x - e^{-x})/(e^x + e^{-x}) \to 1$ as $x \to \infty$. The most remote galaxies in the infinite GR-space are moving at an infinite speed away from the observer while – as described in the finite SR-coordinate system – are moving at the speed of light c away from the observer.

The choice of coordinate system therefore (of course) gives a very much different description of the movement of the galaxy and light as we have seen here.

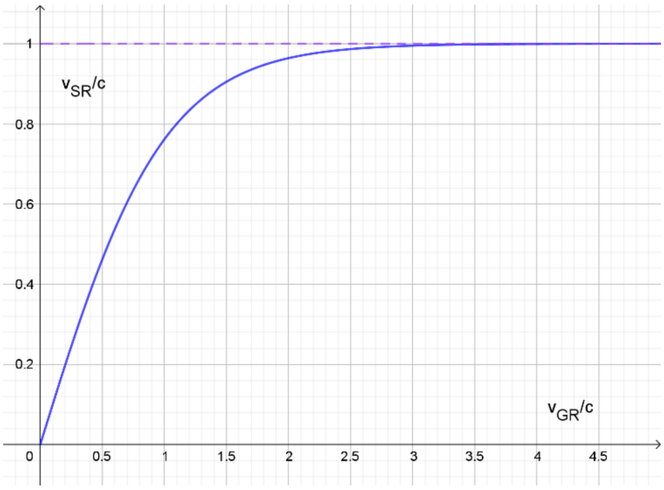

Figure 1: velocity of galaxy in GR og SR - here v_{SR} vs. v_{GR}

If as an example we put $v_{GR} = 2c$ then equation (7) gives the velocity of the galaxy in SR system $v_{SR} = 0.96\ c$. Thus we see that the galaxy is not moving faster than light. The speed of light in GR-coordinates will – using formula (6) – be $v_{GR} = 2c \pm c$, that is $3c$ or $1c$. The proper distance to the observer (us) of the light will grow regardless of the direction of emission (as seen from the galaxy). This is solely a consequence of the choice of coordinate system. If we choose SR-coordinates the velocity of light is $v_{SR} = \pm c$.

In the Milne-universe there are no horizons. We will be able to get information from all parts of the Milne-universe as is the case in SR.

Distances between 'galaxies moving rapidly away from us are in SR-coordinates Lorentz contracted. If we in GR-coordinates have a long line of galaxies separated by a constant mutual distance, the galaxies in SR will not be separated by a constant distance, the more remote galaxies will be separated by a smaller distance because of the Lorentz contraction.

The redshift of light from a galaxy in SR-coordinates is a Doppler shift given by the formula

$$1 + z = \frac{\lambda_0}{\lambda_e} = \sqrt{\frac{1+v_{SR}/c}{1-v_{SR}/c}} \qquad \text{redshift in SR-coordinates}$$

It is easy to show – using formula (7) and formula (8) from below and $R(t) = H_0 \cdot t$ for the Milne Model – that this formula implies the well known formula in GR (the formula below). In more detail formula (8) below gives $s_0 = c/H_0 \cdot \ln(R(t_0)/R(t_e))$ where $R(t_e)$ is the scale factor at the time of emission. Using this result in equation (7) we are led to the wanted result:

$$1 + z = \frac{\lambda_0}{\lambda_e} = \frac{R(t_0)}{R(t_e)} \qquad \text{redshift in GR-coordinates}$$

In Milne-cosmology the cosmological redshift can be viewed upon as a *single* Doppler shift no matter the value of the redshift z. This is a special case of the Milne Model. If the Universe contains matter/energy (as it does!) we can't cover the whole Universe by a single SR-system (inertial system) but must rely on only local SR-systems at rest relative to the Hubble flow (Lorentz systems) so small that the universal gravitational forces has no role when we describe the motions. The size of these systems could be up to 10% of the Hubble length at the time – see below. It is always possible to find such a system as it is assured by the principle of equivalence in GR. The cosmological redshift then can be viewed upon as a continuous Doppler shift of the radiation in its long way from the emitter galaxy to the observer galaxy.

The two formulas of redshift (the SR-formula and the GR-formula) mentioned in this chapter are both correct (the SR-formula only in the Milne Model) but refers to two very different coordinate systems.

The movement of light in the expanding Universe, proper distance and comoving distance

The equation of motion (6) has two solutions when the time t_0 is one of the limits of the integral:

(8) Minus-solution (past): $s(t) = R(t) \cdot \int_t^{t_0} \frac{c \cdot dt}{R(t)}$ and $s_0 = \int_t^{t_0} \frac{c \cdot dt}{R(t)}$

(9) Plus-solution (future): $s(t) = R(t) \cdot \int_{t_0}^{t} \frac{c \cdot dt}{R(t)}$ and $s_0 = \int_{t_0}^{t} \frac{c \cdot dt}{R(t)}$

We have used equation (1) to get the formula for the comoving distance s_0.

Equation (8) gives the proper distance $s(t)$ (and the comoving distance s_0) to the galaxy emitting the light at the time t. We receive the light at the time t_0.

Equation (9) gives the proper distance $s(t)$ (and the comoving distance s_0) to the galaxy receiving the light from us at the time t. We have emitted the light at the time t_0.

In both equations the distance $c \cdot dt$ is being projected to the future/past by using the formula (1) $ds_0 = c \cdot dt / R(t)$.

To calculate those distances we have to know the formula for the scale factor $R(t)$. We will in the next section see how we can solve the cosmological differential equation for this scale factor function.

The cosmological differential equation – how to solve for $R(t)$

The evolution in the scale factor $R(t)$ – and therefore the evolution in the distances to the remote galaxies – is governed by the forces that acts on large scales in the Universe. We will limit ourselves to two kinds of forces. The first is the attracting force between bodies as we know it from the gravitational law of Isaac Newton. This force will slow down the expansion. This force has its root in both the 'normal' matter (baryonic matter) and the so called dark matter (as yet of unknown nature). The second force that influences the movement of galaxies is caused by the so called dark energy and accelerates/pushes the galaxies further apart.

The cosmological differential equation is most simply expressed when we measure the matter/energy densities in units of the so called critical density:

$\rho_{\text{crit}} = \frac{3H_0^2}{8\pi G}$ Critical mass density

H_0 denotes (the present) value of the Hubble constant, and G is the newtonian gravitational constant.

We will use the symbol

$\Omega_m = \frac{\rho_m}{\rho_{\text{crit}}}$ Parameter denoting mass-density of matter

and

$\Omega_\Lambda = \frac{\Lambda}{3H_0^2}$ Parameter denoting density of dark energy

Λ denotes the cosmological constant, describing the energy density of empty (matter-free) space.

We make the remark that the matter parameter Ω_m is a sum of a baryon density parameter $\Omega_B = 0.0490$ (H and He in the early Universe) and $\Omega_C = 0.2607$ being the density parameter of cold dark matter. See ref. 1 (Planck data).

The cosmological differential equation governing the evolution of the scale factor is then

(10) $H_0^{-2} \cdot R''(t) = -0.5 \frac{\Omega_m}{R(t)^2} + \Omega_\Lambda \cdot R(t)$ The cosmological differential equation

where t denotes the (cosmological) time and the primes denotes differentiation with respect to time.

We have disregarded contributions from electromagnetic energy density and from neutrinos. They will play a role in the very early Universe.

The term $-0.5 \frac{\Omega_m}{R(t)^2}$ is the negative 'Newtonian' gravitational part of the acceleration, slowing down the movement of the galaxies relative to each other, while the term $\Omega_\Lambda \cdot R(t)$ describes a positive (repulsive) contribution to the acceleration of the galaxies (if we assume $\Omega_\Lambda > 0$) .

The initial conditions for this differential equation is

(11) $R(t_0) = 1$ og $R'(t_0) = H_0$ Initial conditions

The 'natural' units in the work with cosmological models

$T_H = \frac{1}{H_0}$ Hubble time

and

$L_H = c \cdot T_H = \frac{c}{H_0}$ Hubble length

Measuring the time in units of the Hubble time gives the equation

$R''(t) = -0{,}5 \frac{\Omega_m}{R(t)^2} + \Omega_\Lambda \cdot R(t)$ The cosmological differential equation

And the initial conditions

$R(t_0) = 1$ and $R'(t_0) = 1$ Initial conditions

The unit of velocity is (using the natural units)

$$L_H / T_H = c$$

The equation of motion of the light/photon therefore is

$s'(t) = \pm 1 + \frac{R'(t)}{R(t)} \cdot s(t)$ $s(t_0) = 0$ and $s'(t_0) = \pm 1$

Where we should choose the sign $+$ for light moving away from us and the sign – is chosen when the light is moving against us/the observer.

The two parameters Ω_m og Ω_Λ determines the curvature of space – and therefore weather the space is finite or infinite. To be more precise we should add them together

$\Omega = \Omega_m + \Omega_\Lambda$ Parameter for total mass/energy density

This is the quantity that determines the curvature of space:

a) If $\Omega = 1$ the curvature is 0 and the space is infinite. The geometry of space is Euclidian
b) If $\Omega > 1$ the curvature is positive and the space is finite (2 dimensional model: surface of a ball)
c) If $0 \leq \Omega < 1$ the curvature is negative and the space is infinite

The cosmological parameters have been measured by the e.g. the Planck satellite and the best values are (2018, Planck-satellite data and others):

(12) $$H_0 = (67.66 \pm 0.42)\ {}^{\mathrm{km}}\!/_{\mathrm{s}}\big/_{\mathrm{Mpc}} = (20.74 \pm 0.13)\ {}^{\mathrm{km}}\!/_{\mathrm{s}}\big/_{\mathrm{Mlyr}} = (0.0692 \pm 0.0004)/\mathrm{Gyr}$$

$$\Omega_m = 0.3111 \pm 0.0056$$

$$\Omega_\Lambda = 0.6889 \pm 0.0056$$

Using these values we see that the Universe is very close to the critical value $\Omega = \Omega_m + \Omega_\Lambda = 1$. In accordance with the so called inflation theories. The Universe thus has no (or almost none) curvature and the geometry of space is Euclidian (or almost..). See ref. 1 for more Planck data.

Given the value of the Hubble parameter above we can make the following calculations:

$$T_H = \frac{1}{H_0} = 14.45\ \mathrm{Gyr}$$ Hubble time

$$L_H = c \cdot T_H = \frac{c}{H_0} = 14.45\mathrm{Glyr}$$ Hubble length

The critical density can be evaluated:

$$\rho_{\mathrm{crit}} = \frac{3H_0^2}{8\pi G} = \frac{3\left(2.192\cdot 10^{-18}/\mathrm{s}\right)^2}{8\pi\cdot 6.673\cdot 10^{-11}\mathrm{m}^3/\mathrm{kg}\cdot\mathrm{s}^2} = 8.598 \cdot 10^{-27}\ \mathrm{kg/m^3}$$ Critical density

If we measures this density in terms of H-atoms there will be 5.1 atoms per cubic meter! Are we talking about real nucleons we must 'empty' approximately 4 m^3 to get our hand on 1!

Some exact solutions to the cosmological differential equation

In some cases it is possible to solve the cosmological differential equation exact. We discuss such 3 cases.

a) We can get a whole class of solutions for the critical Universe where $\Omega = \Omega_m + \Omega_\Lambda = 1$:

The solution for the scale factor is

(13) $$R(t) = \left(\sqrt{\frac{\Omega_m}{\Omega_\Lambda}} \cdot \sinh\left(\frac{3}{2} \cdot H_0 \cdot t \cdot \sqrt{\Omega_\Lambda}\right)\right)^{\frac{2}{3}}$$ Scale factor, critical model

The model is often called the Lambda CDM-model (CDM = Cold Dark Matter).
If we wish to find the age of the model universe we have to solve the equation $R(t_0) = 1$. The solution is

(14) $t_0 = \frac{2}{3} \cdot \frac{1}{H_0} \cdot \frac{\ln(\sqrt{\frac{\Omega_\Lambda}{\Omega_m}} + \sqrt{1+\frac{\Omega_\Lambda}{\Omega_m}})}{\sqrt{\Omega_\Lambda}}$ Age, critical universe

Small exercise: derive this formula!
We will use the formula to calculate the age of the Universe using the cosmological parameters from Planck satellite (ref. 1):

$$t_0 = \frac{2}{3} \cdot \frac{1}{H_0} \cdot \frac{\ln\left(\sqrt{\frac{0.6889}{0.3111}} + \sqrt{1 + \frac{0.6889}{0.3111}}\right)}{\sqrt{0.6889}} = 0.954 \cdot \frac{1}{H_0} = 0.954 \cdot 14.45 \text{ Gyr} = 13.79 \text{ Gyr}$$

Which is the best value for the age of our Universe!

We will also answer the question: when did the dark energy 'defeat' the gravitational force of the matter and finally being the driving force in the expansion of the Universe?
To answer this question we take a look at equation (10) – the differential equation of the expansion of the Universe. On the left hand side we have the acceleration of the scale factor. When this acceleration is 0, we have equality between the two forces driving the expansion.
We will solve the equation $R''(t) = 0$:

$0 = -0.5 \frac{\Omega_m}{R(t)^2} + \Omega_\Lambda \cdot R(t)$ Force equality dark energy vs gravitational force

We solve for $R(t)$:

$$R(t) = \sqrt[3]{0.5 \frac{\Omega_m}{\Omega_\Lambda}} = \sqrt[3]{0.5 \frac{0.3111}{0.6889}} = 0.6089$$

where we have used the cosmological parameters from (12). The solution of this equation is

$t = 7.69 \text{ Gyr}$ After this time the dark energy dominates

This already happened $13.79 \text{ Gyr} - 7.69 \text{ Gyr} = 6.1 \text{ Gyr}$ ago.

b) Another exact solution can be found in the case of $\Omega_m = 1$ and $\Omega_\Lambda = 0$ – a critical matter Universe without dark energy:

$R(t) = \left(\frac{3}{2} \cdot H_0 \cdot t\right)^{\frac{2}{3}}$ Scale factor, critical matter model

This model is named the Einstein de Sitter model.
If we again wish to find the age of this model of the universe we have to solve the equation $R(t_0) = 1$. The solution is

$t_0 = \frac{2}{3} \cdot \frac{1}{H_0}$ Age, critical matter universe

This age is 9.63 Gyr if we use the Planck data for the Hubble constant. This age is less than the age of the oldest stars.

c) A third exact solution can be found when $\Omega_m = 0$ and $\Omega_\Lambda = 1$ – a critical dark energy universe without matter.

$$R(t) = e^{H_0 \cdot t}$$ Scale factor, critical dark energy model

This model has no Big Bang because the scale factor never will reach the value 0. It is by the way the only model having a constant Hubble parameter $H(t) = H_0$ for all values of time t.
This model is of course not a realistic model as it contains no matter!

Cosmological distance formulas and the size of the visible Universe

We begin by calculating the size of the visible Universe today, using formula (8):

(15) $$s_{0,max} = R(t_0) \cdot \int_0^{t_0} \frac{c \cdot dt}{R(t)}$$ The size of the visible Universe today

We substitute the scale factor $R(t)$ using (13) and t_0 using (14):

(16) $$s_{0,max} = 1 \cdot \int_0^{\frac{2}{3} \cdot \frac{1}{H_0} \cdot \frac{\ln\left(\sqrt{\frac{\Omega_\Lambda}{\Omega_m}} + \sqrt{1 + \frac{\Omega_\Lambda}{\Omega_m}}\right)}{\sqrt{\Omega_\Lambda}}} \frac{c \cdot dt}{\left(\sqrt{\frac{\Omega_m}{\Omega_\Lambda}} \cdot \sinh\left(\frac{3}{2} \cdot H_0 \cdot t \cdot \sqrt{\Omega_\Lambda}\right)\right)^{\frac{2}{3}}}$$

The cosmological parameters we get from (12) and the result is

$$s_{0,max} = 47.0 \text{ Glyr}$$ Radius visible Universe 2020

This is 3.4 times bigger than the naive value $c \cdot t_0 = 13.79 \text{ Glyr}$. The reason being of course that the distance to the emitting galaxy grows in the time the light travels toward us.

It should be noted that this result does depend somewhat on what we have left out in the early Universe – the radiation energy density and neutrinos. And perhaps most of all: we have not taken into account a possible inflation in the very early Universe! So the value above can be regarded as an approximate size of the 'rediscovered' Universe after the inflation.

Cosmological time dilation

The *cosmological time dilation* can be seen for e.g. in the light curves from supernova type SN1a on cosmological distances. The duration of a typical time evolution will be prolonged by a factor $1 + z$ where z is the redshift of wavelengths in the spectrum of the supernova.

As is the case with distances (like wavelengths) the duration of all physical processes will be longer when they are observed on cosmological distances. You could say that 'video' arriving from far away is shown in slow motion when viewed in the telescope. The duration is prolonged by the factor $1 + z$. If e.g. the redshift is 9, the duration of all physical processes will be prolonged 10 times compared to the original local processes as it is viewed by a local observer.

Another example is the cosmological background radiation having a redshift of approximately 1100, $z \approx 1100$. The time evolution in this radiation will be multiplied by a factor of approximately 1100. The evolution is thus almost brought to a halt compared to the evolution as seen by a local observer – if any were present at the time of emission!

One way to argue for this dilation is this:

We assume that the emitting and receiving galaxy follows the Hubble flow. In this case the comoving distance between the two objects is a constant. Therefore (formula (8))

$$s_0 = \int_{t_e}^{t_0} \frac{c \cdot dt}{R(t)} = \int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{c \cdot dt}{R(t)}$$ comoving distance for the emitting galaxy

The lefthand side of the equation is the comoving distance between the two galaxies traveled by light emitted at the time t_e and received at the time t_0. And the righthand side is the comoving distance between the two galaxies traveled by light emitted at the time $t_e + \Delta t_e$ and receaved at the time $t_0 + \Delta t_0$.

From the equation above it follows (if Δt_e and Δt_0 is short times compared to the cosmological evolution time scale):

$$\frac{c \cdot \Delta t_e}{R(t_e)} = \frac{c \cdot \Delta t_0}{R(t_0)}$$ and thus $$\frac{\Delta t_e}{R(t_e)} = \frac{\Delta t_0}{R(t_0)}$$

Using the relation $1 + z = \frac{R(t_0)}{R(t_e)}$ we finally conclude

$$\Delta t_0 = \Delta t_e \cdot (1 + z)$$ Cosmological time dilation

as claimed. The original timespand Δt_e as measured at the emitting object is 'enhanced' by the factor $(1 + z)$ at the receiver end.

Redshift and Doppler shift

The redshift z is of course defined by the equation

$$1 + z = \frac{\lambda(t_0)}{\lambda(t_e)}$$ Definition of redshift z

where $\lambda(t_0)$ is the measured wavelength in the galaxy spectrum and $\lambda(t_e)$ is the laboratory value of the spectral line as emitted by the remote galaxy.

Can this redshift – no matter how big – be explained using the nonrelativistic Doppler-formula? The answer is – maybe to the surprise of some – yes.

We can argue in this way:

As a start we write the nonrelativistic Doppler formula like this:

$$\frac{\lambda_2 - \lambda_1}{\lambda_1} = \frac{v}{c}$$ Nonrelativistic Doppler formula

λ_1 is the wavelength of light emitted from a source at rest to the observer and λ_2 is the wavelength of the light received by an observer moving away from the source at the velocity v ($v \ll c$). The redshift is in this case given by $z = v/c$.

But why can this 'local' formula be applied to redshifts much bigger than 1? The answer is that it can't – but we can 'chop' the redshift up in minor (local) parts along the path of the light from the emitter galaxy to the receiver galaxy. And then apply the equation on each minor contribution to the redshift.

The emitter galaxy emits – as seen from this galaxy – light of wavelength $\lambda(t_e)$. This light will pass a lot of galaxies/observers on the long way to the observer. We assume that all these galaxies/observers follows the Hubble expansion. The galaxy/observer closest to the emitter galaxy is moving away from the emitter galaxy and the light from the emitter galaxy will – as seen from this next galaxy – have a longer wavelength as the Doppler formula tells us. This is simply physics in a (local) Lorentz frame according to the principle of equivalence. The light moves on towards the next galaxy again receding from the previous galaxy. Redshifting the light further. Etc. (we could also argue that the emitter galaxy is moving backwards away from the next galaxy in line giving rise to a Doppler shift of the wavelengths present in the light from the emitter galaxy).

According to the general theory of relativity it is always possible to choose a local inertial system (Lorentz frame) where we in a limited space can use the laws of physics exactly as in special relativity - without any reference to cosmic gravitational forces represented by the parameters Ω_{m} og Ω_{Λ} (equivalence principle).

The size of such systems should be much less than the Hubble length (at present time 14 Glyr). In such a system the redshift is a non-relativistic Doppler shift due to the movement of the galaxies. The center of this non-expanding non-rotating system in free fall can be chosen to coincide with an observer following the Hubble expansion.

In systems of larger sizes there will be second order effects when we want to describe movements – a sign that tells us that the cosmological gravitational forces can no longer be ignored.

We transform the Doppler formula from above in the following way

$$\frac{\lambda_2-\lambda_1}{\lambda_1}=\frac{v}{c} \qquad \text{transform to:} \qquad \frac{\Delta\lambda}{\lambda}=\frac{v}{c}$$

The velocity v is given by the Hubble law:

$$\frac{\Delta\lambda}{\lambda}=\frac{v}{c}=\frac{H(t)\cdot s}{c}=H(t)\cdot\frac{s}{c}=H(t)\cdot\Delta t$$

The time $\Delta t = s/c$ is the time for the light to move from one nearby galaxy to the next in the path of the light from emitter galaxy to observer galaxy.

But we remember the definition $H(t) = R'(t)/R(t)$ and therefore

$$\frac{\Delta\lambda}{\lambda}=H(t)\cdot\Delta t=\frac{R'(t)}{R(t)}\cdot\Delta t=\frac{R'(t)\cdot\Delta t}{R(t)}=\frac{\Delta R}{R}$$

where we have used the approximation $R'(t)\cdot\Delta t = \Delta R$ valid for small values of Δt.

Thus we have

$$\frac{\Delta\lambda}{\lambda}=\frac{\Delta R}{R}$$

We now change to differentials and make an integration on both sides:

$$\int_{\lambda_e}^{\lambda_0}\frac{d\lambda}{\lambda}=\int_{R(t_e)}^{R(t_0)}\frac{dR}{R}$$

from which we get

$$\ln\left(\frac{\lambda_0}{\lambda_e}\right)=\ln(\frac{R(t_0)}{R(t_e)})$$

We remove the ln-function on both sides and finally we have

(18) $$1+z=\frac{\lambda_0}{\lambda_e}=\frac{R(t_0)}{R(t_e)}$$ Doppler shift: Doppler upon Doppler upon…

The wavelength scales as the scale factor (and therefore also as the distances) when we follow the continuous Doppler shift happening as the light/photons moves through space towards us.

Actually we could have argued for this in an even simpler way without referring to the Hubble law:

we again as a starting point looks at the non-relativistic Doppler shift formula from above:

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c}$$

We multiply both v and c by Δt – the time the light uses to pass from one galaxy to the next (nearby) galaxy that is moving away from us at the velocity v:

$$\frac{\Delta\lambda}{\lambda} = \frac{v \cdot \Delta t}{c \cdot \Delta t}$$

The distance between the galaxies is $s = c \cdot \Delta t$ and the distance between the galaxies has increased by $\Delta s = v \cdot \Delta t$. Therefore we can conclude:

$$\frac{\Delta\lambda}{\lambda} = \frac{\Delta s}{s}$$

The relative growth in wavelength is equal to the relative growth in distance. From this we conclude (as argued above) that the wavelength and the distance between the galaxies are proportional:

$$1 + z = \frac{\lambda_0}{\lambda_e} = \frac{s(t_0)}{s(t_e)}$$

But the distances are proportional to the scale factors according to equation (1). We therefore end up having shown equation (18) for the redshift once again. See ref. 5.

In the Milne universe where we can apply both GR-coordinates and SR-coordinates to the whole Universe it is evident that even the largest redshift can be viewed as a single Doppler shift. As we argued in the section relating to the Milne model. See ref. 7.

Pedagogical advantages of the Doppler explanation

As argued above there is no reason to implement a special principle in cosmology a la 'the space is expanding – and the waves expands with it' if we want to derive the redshift formula (18). We only need the non relativistic Doppler formula – with repeated usage along the path of the light from emitter galaxy to observer galaxy. Actually it may be more correct to say that the light waves are stretched continuously by Doppler shifts as the distances expands.

Using our reasoning from above we can avoid sentences like: 'the galaxies are not really moving – even though the distances between then are growing. It is just the space between them that is expanding'

We have though introduced a coordinate system where galaxies following the Hubble flow have non-changing coordinates. But the reason for these non-changing coordinates is that these coordinates are comoving coordinates! Not surprising they do not change. The time dependence of the distances is contained in the scale factor $R(t)$ in equation (1). It means that the galaxies are actually moving when we use the proper distance as the coordinate of the galaxy. And the galaxies are in free fall in the cosmological gravitational field.

You have been driving too fast in your car and the police have measured your speed using a Doppler laser gun. The laser gun showed that your speed was 120 km/h but you were allowed only to drive at the speed of 50 km/h. You tell the police: 'It was not my car that moved too fast – it was the space between the laser gun and the car that expanded too fast. My car has a non-changing commoving coordinate as you can see – I have painted it myself on the side of the car – it reads 17 km. No change in my commoving position. Therefore I cannot accept the fine. I was not moving at all'. The police will probably give you an extra fine for trying to explain away your crime. The movement was real – even though your commoving coordinate did not change.

In the general theory of relativity it is the matter and its movements that generates the gravitational fields and thereby the geometry of spacetime. And the geometry of spacetime 'rules' the movement of the matter. The space is not an independent actor dragging the matter along, as the balloon model could imply.

Calculations shows that a galaxy not following the Hubble flow will not just join the Hubble flow immediately. The galaxies are in free fall in the cosmic gravitational field no matter whether they follow the Hubble flow or not. Newton rules. See ref. 8.

- " This (the examples in the article) have proved that 'expanding space' is in general a dangerously flawed way of thinking about an expanding universe" - Peacock 2010

This does not mean that there is a discrepancy between the Doppler explanation and 'space expanding' explanation when it comes to explaining the redshift equation (18) as the 'space expand' explanation is introduced to 'explain' this equation. But pedagogically it is unfortunate to talk about movements that are not real movements. And also unfortunate to introduce a 'space expands' explanation to explain the redshift equation (18) when it is not necessary. See ref. 3, 4, 7, 8, 10.

Using the 'space expands' explanation to 'explain' the redshift equation (18) also gives you the problem of why you can use the Doppler formula if the redshift is small.

Another advantage of using the Doppler explanation is that questions like 'if space is expanding why is the earth not expanding too? And what about the atoms? The Solar System? The Milky Way?' These questions will not naturally pop up when you use the Doppler explanation.

In more textbooks of physics you see a description of the cosmological redshift divided in two. In the nearest part of the Universe the Doppler explanation involving real movements of the galaxies is applied. But on larger scales it's the space that is expanding (and not the galaxies that are moving – they are carried away by space expansion).

To this description you may ask: at what redshift does the space expansion take over?

A more serious consideration to this description could be: we send a series of *cosmic explorers* to the galaxies where the light has passed at an earlier time. These explorers ask the local observers along the path of the light how they did experience the redshift of the passing light to find out whether they have experienced that space was expanding. And everywhere they get the answer: we saw a Doppler shift in our vicinity. In the same way that we on earth sees the redshifts in our vicinity. And this observation is by the way in accordance with the cosmological principle: the expansion looks the same no matter from where you experiences it. If the expansion gives rise to a Doppler shift on short distances the explanation is the same on every part of the path of the light. So we are led to the view that the redshift is a series of Doppler shifts – Doppler upon Doppler upon…

The 'blowing up a balloon' model and the 'raising raisin bread dough' model of the expansion of the Universe should not be taken too literally – the 'medium' does not 'pull' the 'galaxies along as a kind of new ether. In the Universe the galaxies (or maybe more correct: groups of galaxies) are in free fall in the cosmic gravitational field governing the movements. But the models *are* usable in explaining some aspects of the expansion. As e.g. there is no center of the expansion (even though the balloon model could be misleading here), usable in illustrating of the Hubble law and usable in illustrating how the waves are stretched in the expansion - even though the models does not give an explanation of why the galaxies moves as they do or how the redshift of the wavelengths happens.

Ref. 1: https://arxiv.org/abs/1807.06205

Ref. 2: https://www.mso.anu.edu.au/~charley/papers/LineweaverEganParisv2.pdf

Ref. 3: http://www.astro.ucla.edu/~wright/cosmology_faq.html

Ref. 4: http://math.ucr.edu/home/baez/physics/Relativity/GR/hubble.html

Ref. 5: Steven Weinberg: The first 3 minutes, 1979 chapter 2

Ref. 6: https://youtu.be/DClEXO0pCZ4 - Stephen Hawking - expanding Universe and Dopplereffect

Ref. 7: https://arxiv.org/pdf/0808.1081.pdf - redshift as a kinematic effect – the galaxies are moving!

Ref. 8: http://indico.ictp.it/event/a09159/session/2/contribution/1/material/0/0.pdf

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Ref. 9: http://www.jb.man.ac.uk/distance/frontiers/cosmology/node2.htm - Cosmological expansion and redshift

Ref. 10: https://www.nbi.ku.dk/spoerg_om_fysik/astrofysik/universetsudvidelse/ (Danish website)

Ref. 11: Milne model http://www.chronon.org/articles/milne_cosmology.html