Børge Leo Physics 2020

Beta decay of ^{137}Cs and the following decay of ^{137}Ba^{m}

Many teachers and students have in the past made the laboratory-exercise of determining the half-life of the gamma source Ba-137* created by the beta-decay of Cs-137. The gamma-energy of the excited state in Ba-137* - when decaying to the ground state – is 0,663 MeV. This excited state is called metastable or isomeric because of the relative long half-life.
Can we explain why this half-life is so long? How can we explain the decay in terms of nuclear orbitals? And finally: how can we characterize the beta-decay(s) of Cs-137 in terms of more or less forbidden decays? We will seek the answers to these questions below.

W begin by make some remarks on the single-particle model for the atomic nucleus. In this model the nucleons are moving in a mean-potential created by the other nucleons. Neutrons and protons are moving in separate potentials, the difference mainly caused by the Coulomb-repulsion among the protons.
The nucleons are fermions (spin $1/2$-particles), and in every spatial orbit there are at maximum two nucleons of same type, one spin up, and one spin down.
If the nucleus has an uneven number of neutrons and an even number of protons (as is the case for Ba-137), spin and parity of the nucleus as a whole is determined by the ‘unpaired’ neutron. More on this topic later.
The decay equation for the beta decay is either

$$^{137}_{55}Cs \rightarrow {}^{137}_{56}Ba^{*} + {}^{0}_{-1}e + \bar{\nu}_e$$

followed by $$^{137}_{56}Ba^{*} \rightarrow {}^{137}_{56}Ba + \gamma$$

or alternatively

$$^{137}_{55}Cs \rightarrow {}^{137}_{56}Ba + {}^{0}_{-1}e + \bar{\nu}_e$$

where the barium nucleus is created directly in its ground state.

The decay scheme of the γ-decay:

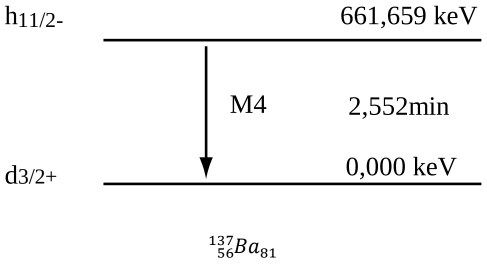

Fig. 1: Decay scheme for $^{137}Ba^{m}$

with spin and parity of initial- and final states shown, together with the energy of the states, half-life of the decay and the type of EM transition that is happening. Here a magnetic transition of order 4 (which means, that the gamma-photon carries of an angular momentum of $4\hbar$ in the decay)

We will make some remarks on the nuclear states and the value of the half-life below.

Nuclear shell structure elements – the single particle model

But what has the decay scheme to do with the nuclear structure? The values of spin and parity for the initial- and final states are of course nuclear states in the Ba-137 nucleus, and can essentially be described as an excited ($h_{11/2}$-) neutron state and the ground neutron state ($d_{3/2}$+). The excited ($h_{11/2}$-) state is a neutron hole in the $h_{11/2}$- orbital, see below. A neutron from this orbital has decayed to a proton in the beta decay and has left the neutron hole. The ground state $d_{3/2}$+ can be described as a neutron hole in the $d_{3/2}$+ orbital.
The cesium-137 nucleus has 82 neutrons and 55 protons, whereas the barium-137 nucleus has 81 neutrons and 56 protons.

The neutron number 82 is one of the so-called magic numbers, which gives especially stable neutron/proton configurations, somewhat like the noble gasses when we think of atomic structure. The spherical single particle model is only useful for nuclei where the neutron/proton numbers are relative close to the magic numbers 2, 8, 20, 28, 40, 50, 82, 126. If we are not close to one of these numbers, the nucleus is not spherical!
Fig. 2 shows an energy diagram of neutron-orbitals for nuclei with neutron numbers between the magic numbers 50 and 82. In the case of the nucleus Cs-137 all these orbitals are filled.

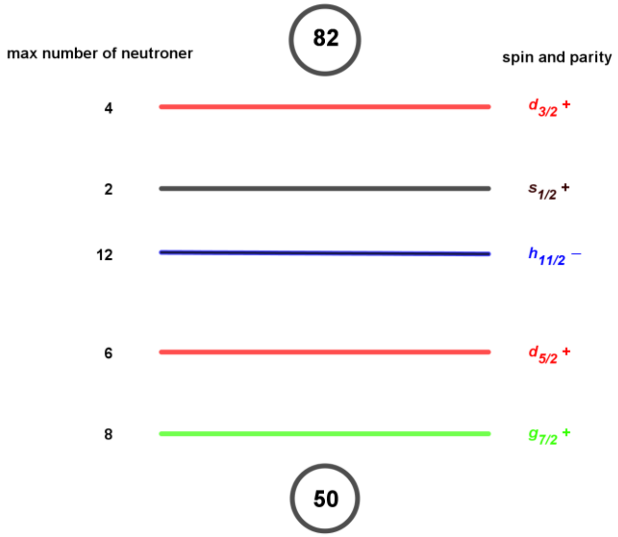

Fig. 2: neutron orbitals in nuclei with neutron numbers from 50 to 82

In the beta-minus decay of Cs-137 one of the neutrons in the orbitals $h_{11/2}$, $s_{1/2}$ or $d_{3/2}$ decays (and a proton, electron and anti-electron neutrino are created), and a neutron hole is created in one of these orbitals. In the case of the orbitals $h_{11/2}$ or $s_{1/2}$ a neutron from the $d_{3/2}$-orbital will 'drop down' in the neutron hole, and a gamma photon will (in most cases) be emitted.

If the neutron hole is created in the $d_{3/2}$-orbital, the Ba-137 nucleus is created in its ground state.

These neutron holes has the same spin and parity as the orbital containing the hole, because the neutron number of the orbital is un-even – and therefore there is an un-paired neutron in the orbital, and this un-paired neutron determines the spin and parity of the nucleus (no protons are unpaired because the proton number is even, 56).

You may be asking why it is neutrons from the $h_{11/2}$ -state, $s_{1/2}$ -state or $d_{3/2}$ -state that decays to a proton in the beta-minus decay. The answer is that these neutron orbitals have higher energy than the $g_{7/2}$ –proton state, where the newly created proton ends. After all the Cs-137 is a product of a fission process and therefore contains 'too many' neutrons to be stable.

Estimate of half-life for Ba-137*

The decay of the $h_{11/2}$ –state to the $d_{3/2}$ –state creates a photon carrying off the angular momentum $11/2\hbar - 3/2\hbar = 4\hbar$ - a high value. And the nature of the decay is magnetic (the neutron is a small magnet), actually a M4-transition involving parity-shift. These two facts – and the fact that the transition-energy is relatively small – explains the relative small decay-constant and therefore also the relative long half-life of the $h_{11/2}$ –state.

An estimate of the decay constants for electric transitions E1, E2, … and magnetic transitions M1, M2, … are given by the so-called Weisskopf estimates for single particle transitions in table 1. If we use this on the actual decay we find the decay-constant

$$k = 3{,}3 \cdot 10^{-6} A^2 \cdot E^9 = 3{,}3 \cdot 10^{-6} 137^2 \cdot 0{,}662^9 \text{ s}^{-1} = 0{,}0015 \text{ s}^{-1}$$

and therefore the estimate of half-life is $T_{\frac{1}{2}} = \frac{\ln(2)}{k} = 460 \text{ s} = 7{,}6 \text{ min}$

compared to the value 2,552 min from experiment. So the estimate is fair!

As a comparison: if the transition had been an E2-singleparticle – which by the way cannot give a spin-change of 4 - then the half-time would be $4{,}8 \cdot 10^{-8}$ sec, given the same transition-energy.

The small value of the decay constant are partly due to the big difference in the spins of initial- and final-states. The radial wavefunction for the high spin state is more ’concentrated’ near the edge of the nucleus than the radial wavefunction for the low spin state, making the matrix-element for the transition small.

Long-lived isomeric states like the one in ^{137}Ba^{m} is also found in ^{139}Ce, half-time $T_{\frac{1}{2}}$ = 54,8 s, transition-energy E_γ = 0,754 MeV and in ^{135}Xe, $T_{\frac{1}{2}}$ = 15,3 min, E_γ = 0,527 MeV … – all happening as M4-transitions. You can check for yourself how well the estimate in table 1 agrees with these experimental values of half-life. The initial and final states have same spin-parity values as in the ^{137}Ba^{m} decay.

Transition	Name of transition	Angular momentum L_{foton} ($\hbar$)	Parity-shift	k in s^{-1}
E1	Electric dipole	1	Yes	$1{,}0 \cdot 10^{14} A^{\frac{2}{3}} E^3$
M1	Magnetic dipole	1	No	$3{,}1 \cdot 10^{13} E^3$
E2	Electric quadrupole	2	No	$7{,}4 \cdot 10^{7} A^{\frac{4}{3}} E^5$
M2	Magnetic quadrupole	2	Yes	$2{,}2 \cdot 10^{7} A^{\frac{2}{3}} E^5$
E3	Electric octupole	3	Yes	$3{,}5 \cdot 10^{1} A^{2} E^7$
M3	Magnetic octupole	3	No	$1{,}1 \cdot 10^{1} A^{\frac{4}{3}} E^7$
E4	Electric hexadecapole	4	No	$1{,}1 \cdot 10^{-5} A^{\frac{8}{3}} E^9$
M4	Magnetic hexadecapole	4	Yes	$3{,}3 \cdot 10^{-6} A^{2} E^9$

Tabel 1. Weisskopf estimates of decay constant k for different multipole-orders L as function of mass number A and transition energy E in MeV.

Beta decay of Cs-137 – and other nuclei

Fig. 3 below gives a quite detailed view of the decay. Further details on the decay will be mentioned below.

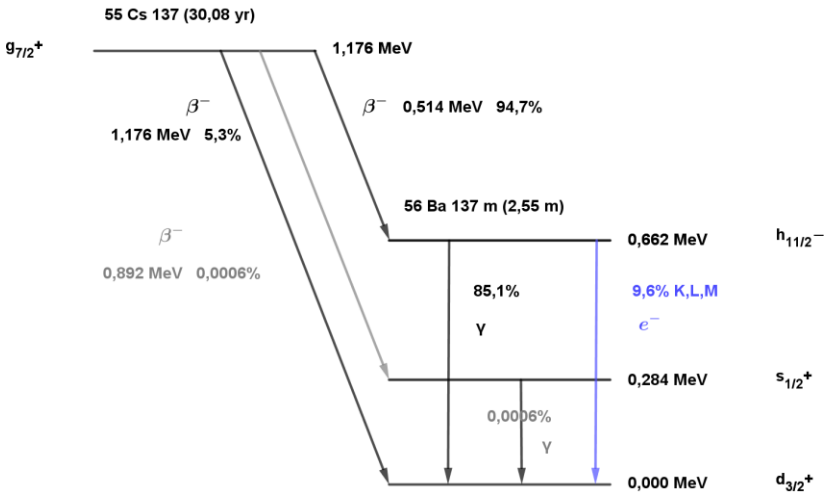

Fig. 3: decay scheme for Cs-137

As shown in fig. 3 94,7% of beta-minus decays ends in the exited state $h_{11/2}$-, and 5,3% goes to the ground state $d_{3/2+}$ of the Ba-137 nucleus. And a very small fraction in the $s_{1/2+}$ – state.

But not all of the 94,7% beta-decays ending in the $h_{11/2}$- state decays to the ground state by gamma emission. Actually 9,6% (percentage of beta-minus decays) decays by IC – internal conversion where the energy surplus is transferred to an electron the inner shells of the atom, the gamma energy given to the electron is though reduced by the binding energy of electrons in those shells. The other decays from the $h_{11/2}$- state (85,1% of the beta decays) happens by emission of a photon.

Actually the decay of the $h_{11/2}$- state to the ground state also can happen as a 2 gamma process (not shown in figure). But only in a very small fraction (10^{-4}%).

Allowed and forbidden decays

Now back to fig. 3. Can we understand in some details the differences in the decay rates of the 3 beta decays shown in the figure? The answer is yes.

To understand the differences in decay rates we must – among other things - focus on the change in spin value from initial nuclear state to final nuclear state. This change in spin value has to be delivered by the $(e^-, \bar{\nu}_e)$-pair. These two particles are both fermions each with internal spin value ½. Combined they can deliver spin values of 0 or 1.
But this is not enough to explain the spin differences from the initial $g_{7/2+}$ state to the final states shown in fig. 3. How then can the decay happen? The answer is of course that we have forgot the (total) orbital angular momentum of the $(e^-, \bar{\nu}_e)$-pair. If we use the letter L for this quantity, then we have the possible values $L = 0, 1, 2, \ldots$ in units of $\hbar$.
If $L = 0$ we have the so-called *allowed* decays and if $L > 0$ we have the so-called *forbidden* decays. The allowed transitions have a much higher decay rate (decay constant) than the forbidden transitions. And a higher value of L will make the decay rate for the transition much smaller, resulting in a much bigger half-life.
How can this be explained? We here try to give a classical mental picture – even though the real explanation is of course quantum-mechanical.
In this picture a value of $L > 0$ means that the $(e^-, \bar{\nu}_e)$-pair is 'shot out' off-center compared to the nucleus as compared to the $L = 0$ case ('head on'). In quantum-mechanical terms it means the overlap of the $(e^-, \bar{\nu}_e)$-wavefunction and neutron-wavefunctions is much smaller in the case $L > 0$ than in the case $L = 0$.
We begin on our classical picture by calculating the radius of the nucleus:

$$r_{nucleus} = 1{,}25\ \text{fm} \cdot A^{\frac{1}{3}} = 1{,}25\ \text{fm} \cdot (137)^{\frac{1}{3}} = 6{,}4\ \text{fm}$$

where A is the mass number of the nucleus. We now look at the decay of the Cs-137 nucleus to the excited state $h_{11/2}$- of the Ba-137 nucleus with a Q^*-value of 0,514 MeV. We further assume that

the electron has the velocity 0 which means that the (anti)neutrino gets the whole Q^*-value. We neglect the mass of the neutrino.
We then calculate the momentum of the neutrino p from the equation

$$E^2 = p^2 \cdot c^2 + m^2 \cdot c^4 \qquad \text{energy } E\text{, momentum } p \text{ and mass } m$$

And the result is

$$p = \sqrt{\frac{E^2}{c^2} - m^2 \cdot c^2} = \sqrt{\frac{(0{,}514\ \mathrm{MeV})^2}{c^2} - 0^2} = \frac{0{,}514\ \mathrm{MeV}}{c}$$

We further assume that the neutrino is ’shot off’ in the distance b from the nuclear center the momentum being orthogonal on b.

The angular momentum L is then given by the equation

$$L = b \cdot p$$

And we find (if we assume $L = 1\hbar$)

$$b = \frac{L}{p} = \frac{1\hbar}{0{,}514\ \mathrm{MeV}/c} = \frac{1\hbar c}{0{,}514\ \mathrm{MeV}} = \frac{197{,}327\ \mathrm{MeV} \cdot \mathrm{fm}}{0{,}514\ \mathrm{MeV}} = 384\ \mathrm{fm}$$

Where we have used the value $\hbar \cdot c = 197{,}327\ \mathrm{MeV} \cdot \mathrm{fm}$.

In this classical picture the $(e^-, \bar{\nu}_e)$-pair is created in the distance of 384 fm from the nuclear center, ’miles’ from the nucleus! So in the classical picture there is no connection between the nucleus and the $(e^-, \bar{\nu}_e)$-pair. In quantum mechanical terms this means that the overlap between the $(e^-, \bar{\nu}_e)$-wavefunction and the neutron wavefunctions in the nucleus must be very small given

the size of the nucleus is only 6,4 fm. And the decay constant of this beta decay must therefore very small.
We emphasize that the purpose of this simple classical calculation is mostly mental: it gives a simple way to ‘understand’ the small decay rates for the ‘forbidden’ decays.

We remark that an orbital angular momentum of $L = 1\hbar$ in quantum mechanics is described by the spherical harmonics $Y_{1,m}$, where $m = -1, 0, 1$. These functions changes sign in a space reflection and therefore implies a parity change between initial- and final state of the nucleus.

The beta minus decays of Cs-137 to states in Ba-137 – the details

$g_{7/2+} \rightarrow h_{11/2-}$: 94,7%
The transition from the $g_{7/2}+$ state in Cs-137 to the excited state $h_{11/2}-$ in Ba-137 is an $L = 1$ forbidden transition (unique) where the internal spins of the $(e^-, \bar{\nu}_e)$-pair are coupled to spin 1. And this spin is aligned with the orbital angular momentum to give a spin change between initial- and final states of 2 units, in agreement with the calculation $\frac{11}{2} - \frac{7}{2} = 2$. Furthermore there is a change in parity from initial state to final state in agreement with the orbital angular momentum $L = 1$ of the $(e^-, \bar{\nu}_e)$-pair.

$g_{7/2+} \rightarrow s_{1/2+}$: 0,0006%
The transition from the $g_{7/2}+$ state in Cs-137 to the excited state $s_{1/2}+$ in Ba-137 is an $L = 2$ forbidden transition (unique) where the internal spins of the $(e^-, \bar{\nu}_e)$-pair are coupled to spin 1. These two spins are aligned to give the spin change between initial and final states of 3 units in agreement with the calculation $\frac{7}{2} - \frac{1}{2} = 3$. There is no change of parity in agreement with $L = 2$. This transition has a much lower decay rate than the previous transition because of the higher L-value.

$g_{7/2+} \rightarrow d_{3/2+}$: 5,3%
The transition from the $g_{7/2}+$ state in Cs-137 to the ground state $d_{3/2}+$ in Ba-137 is an $L = 2$ forbidden transition where the internal spins of the $(e^-, \bar{\nu}_e)$-pair are *not* aligned with the orbital angular momentum of the pair. The total spin of the $(e^-, \bar{\nu}_e)$-pair is 2, which also is the difference of the spins of initial- and final states $\frac{7}{2} - \frac{3}{2} = 2$. This non alignment of the spins means that more nuclear matrix elements are involved, giving more phase space for the $(e^-, \bar{\nu}_e)$-pair and therefore gives a higher transition rate than if the spins are aligned. This transition has therefore a higher transition rate than $g_{7/2+} \rightarrow s_{1/2+}$ transition.

Forbidden transition – another example: from In-115 to Sn-115

The process is

$$^{115}_{49}In \rightarrow {}^{115}_{50}Sn + {}^{0}_{-1}e + \bar{\nu}_e$$

The Q-value of this process is 0,49749 MeV.

The In-115 nucleus has a ground state $g_{9/2+}$ (proton), og the Sn-115 nucleus has a ground state $s_{1/2+}$ (neutron). The spin change is 4 units(!) and the transition is classified as forbidden with $L = 4$. This high degree of forbiddennes gives a long half-time, that is $4{,}41 \cdot 10^{14}$ years – which is app. 32000 times the present age of the Universe(!)

Speaking of another long half-life we can think about the nucleus Bi-209. It will decay in an alfa-transition with half-time $2{,}0 \cdot 10^{19}$ years – or app. 1,5 billion times the present age of the Universe(!!). If you've got 1 gram of this metal you will experience app. 100 alfa-decays in a year. Not the most dangerous radioactive material you may have in your lab. But this is another story. See Ref. 9

Table 2: different transition types of beta-minus decays

Type of transition	Order of forbiddenness L	Difference, nuclear spin Δj	Parity-change
Allowed, superallowed	0	$0, \pm 1$	no
Forbidden, unique* ($L > 0$) Elektron- and neutrino-spin aligned with orbital angular momentum L	1 2 3 4 …	± 2 ± 3 ± 4 ± 5 …	yes no yes no …
Forbidden, non unique ($L > 0$)	1 2 3 4 …	$0, \pm 1$ $\pm 1, \pm 2$ $\pm 2, \pm 3$ $\pm 3, \pm 4$ …	yes no yes no …

*: in essence only one nuclear matrix element is involved

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Ref.1: tilladte og forbudte beta-henfald i astrofysikken – konsekvenser for stjerners slutfaser (danish)
https://aktuelnaturvidenskab.dk/find-artikel/nyeste-numre/4-2020/stjerners-endeligt/

Ref.2: A Forbidden Transition Allowed for Stars
https://physics.aps.org/articles/v12/151
Ref.3: 2 gamma-transitions https://www.eli-np.ro/science_article.php?id=4
Ref.4: Electromagnetic transition rates
http://oregonstate.edu/instruct/ch374/ch418518/CHAPTER%209%20GAMMA%20RAY%20DECAY-rev.pdf
Ref.5: Rules for beta-minus decay etc
http://www.umich.edu/~ners311/CourseLibrary/bookchapter15.pdf
Ref.6: https://www.scribd.com/presentation/397693881/Bucharest-Kondev-Logft
Ref.7: non-SI-units https://physics.nist.gov/cuu/pdf/nonsi_2006.pdf
Ref.8: Nuclear shell model http://ne.phys.kyushu-u.ac.jp/seminar/MicroWorld3_E/3Part2_E/3P26_E/shell_model_E.htm
Ref.9: decays with long half-life
https://epja.epj.org/articles/epja/abs/2019/08/10050_2019_Article_12823/10050_2019_Article_12823.html
Ref.10: experimental test of neutron holes
https://www.researchgate.net/publication/258667574_Neutron-hole_strength_in_the_N_81_isotones/link/5728cca608ae057b0a033714/download