Elements of the Milne Model of the Universe

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The Milne Model was created by Edward Arthur Milne in the year of 1935. The Model is – in GR-coordinates (see below) – consistent with the principles of homogeneity and isotropy – also called the Cosmological Principle.

In the Milne Model there are no gravitational forces, and therefore the velocities of the 'galaxies' are constant. The absent of gravitational forces makes it possible to describe this Universe not only by the usual coordinate system involving proper time and proper distance (in this article called a GR system), but also a special relativistic coordinate system (called a SR system)

Even though this model does not correspond to the actual Universe for obvious reasons, this simple model never the less gives a valuable insight in the somewhat strange properties of the coordinate system most often used in cosmology. This involves for example the apparent faster-than-light velocities for galaxies – a feature of all models involving an infinite Universe and where the Hubble law is valid.

In the Milne Model we can translate the velocity of the galaxies in GR coordinates directly to the more well known special relativistic coordinate system (here also called a Minkowskian system or SR), where the maximum velocity is the (constant) speed of light usually named by the letter c. And in this way we can answer the question: does the galaxy really move faster than light?

As there are no gravitational forces in the model, we can describe the whole Universe in two ways:

1) By a general relativistic model (GR) where the distances in the Universe are proper-distances, i.e. the distance between two galaxies A and B at a given time is the sum of locally defined proper-distances/rest-distances between close pairs of galaxies that is all following the Hubble flow. The time is the proper time for each galaxy since Big Bang, following the Hubble flow. In this system the Hubble law is valid, cf. fig. 1 below.
2) By a special relativistic model (SR) where the whole Universe is covered by a single Minkowskian system. In this description the Universe is finite, limited by the maximum distance from the observer given by $x = c \cdot t$, where t is the age of the Universe, cf. fig. 2 below.

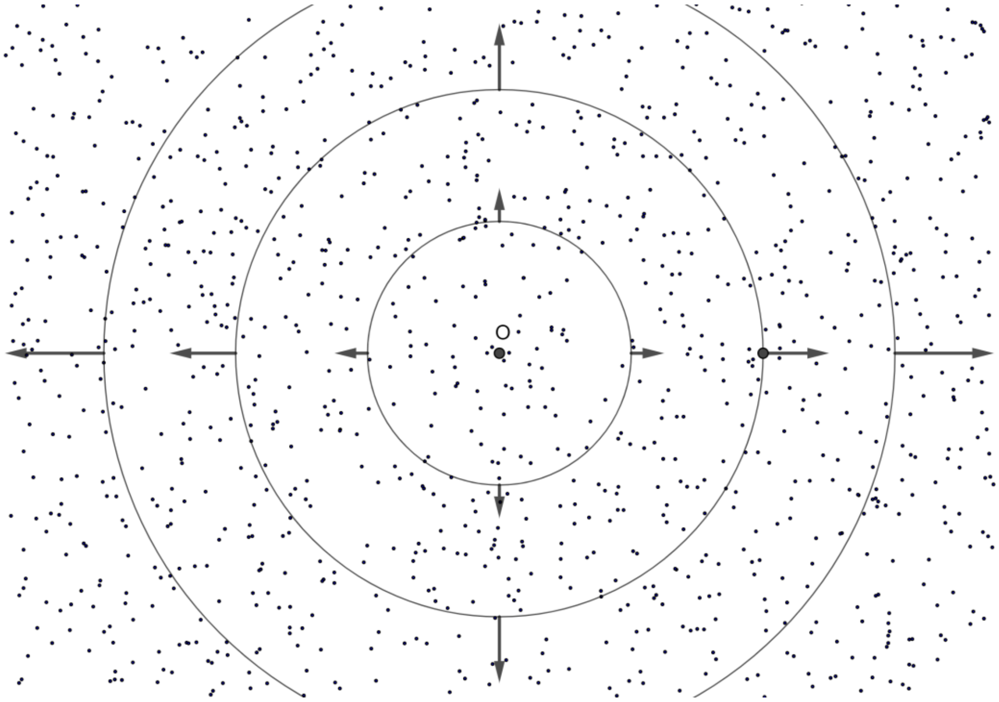

Fig. 1: The Milne model i GR-coordinates. The Universe is infinite, the 'galaxy' density is constant in space, and the Hubble law is valid.

The Milne model (GR) is the Minkowskian Frame of reference (SR) translated to an expanding reference frame of reference.

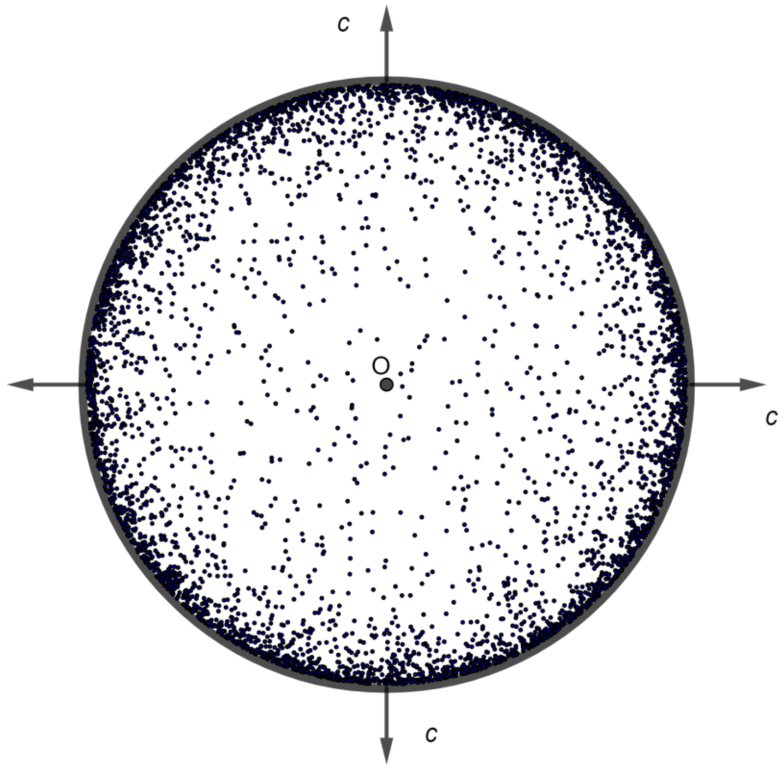

Fig. 2: The Milne-model in SR-coordinates (Minkowski-system). The Universe is finite, limited by the radius x = c·t, where t is the Age of the Universe and c is the speed of light. The density of 'galaxies' is increasing towards the edge of the Universe and is greater than in the GR-model except at the point of the observer at the point O.

The density of galaxies n_τ in the GR-model is constant in space, but (of course) decreasing with time. The density of galaxies in the SR-model n_t is also decreasing with time, but rising towards infinity when x – the distance to the observer – approaches the radius of the Universe $c{\cdot}t$:

$$n_t = \frac{n_\tau}{\sqrt{1-\left(\frac{x}{c\cdot t}\right)^2}}$$ The connection between density of galaxies in the two models

Check ref. 4 for more details.

The description of the Universe in the Minkowskian frame is isotropic, but not homogeneous as the density of galaxies increases toward the edge og the Universe.

By using the transformation between the two systems (see below) it is possible to show that even galaxies having a high value of the redshift z (ex. $z = 10$) does not move faster than light even though the speed of the galaxy in GR-coordinates exceeds the value c. In the SR-system the speed of light is precisely c and the speed of material bodies cannot exceed this value.

We are also able to show, that when we receive light from galaxies having high values of the redshift, the redshift can always be thought of as a series of small Doppler shifts, or to be more precise: the high values of redshifts can be thought of as a continuous Doppler shift of the light on its long way to our telescopes when it passes from one galaxy/observer to a next nearby galaxy/observer that is also following the Hubble flow and therefore is moving away from the former – giving rise to a small redshift. This process is happening all the time on the long journey towards our telescopes.

This is valid not only for the Milne Model. Even though we cannot cover the whole Universe by a single Minkowski system when there are gravitational forces to take into account, it is on smaller scales of space and time always possible to implement a local Minkowski system in which it is possible to describe the process of redshift as a Doppler Shift. That's part of the dna of the General Theory of Relativity.

The transformation between the two systems GR and SR will give us insight into the connection between GR-velocity and SR-velocity, and we will see that - no matter the size of the redshift of a galaxy - the galaxy does not move faster than light. We simply transform the GR velocity (with no upper limit) to the corresponding SR velocity.

In this article, the 'dots' on the two figures 1 and 2 are called galaxies – even though they are massless!

The two systems and the transformation between them

Now to the two systems and the transformation between them. We will only consider radial movements relative to the observer.

The GR-system: we (the observer) are placed in the point O ('the Milky Way') At the proper time τ_0 ('now') we will denounce the proper distance to another galaxy (following the Hubble Flow) by s_0. At the proper time τ the proper distance to this galaxy is

(1) $s(\tau) = a(\tau) \cdot s_0$ proper distances scales by the factor $a(\tau)$

In this equation $a(\tau)$ is the scale factor, which in the Milne Model is particularly simple because there are no gravitational forces to accelerate the galaxies:

(2) $a(\tau) = k \cdot \tau = \tau/\tau_0$ Scale factor in the Milne Model

The galaxies are receding from the observer at the Point O by a constant velocity:

(3) $v_{GR} = \frac{s(\tau)}{\tau} = \frac{s_0}{\tau_0} = k \cdot s_0$ GR velocity of galaxy

As there are no upper limit to the distance s_0, there also are no upper limit to the GR velocity of the galaxy.

But how do the description of the movement of the galaxy look in GR coordinates?

We (the Observer at the Point O) are sending a short light pulse from our position at O at the time τ_0. The light pulse reaches the remote galaxy at the time τ, are reflected and returns to the observer at the point O at the time τ_1 – as shown in fig. 3. In fig. 3 (parts of) the light pulse is reflected from two different galaxies gal1 and gal2. The reflection point and time of the light hitting the galaxy defines a point on the galaxy (τ, s) graph.

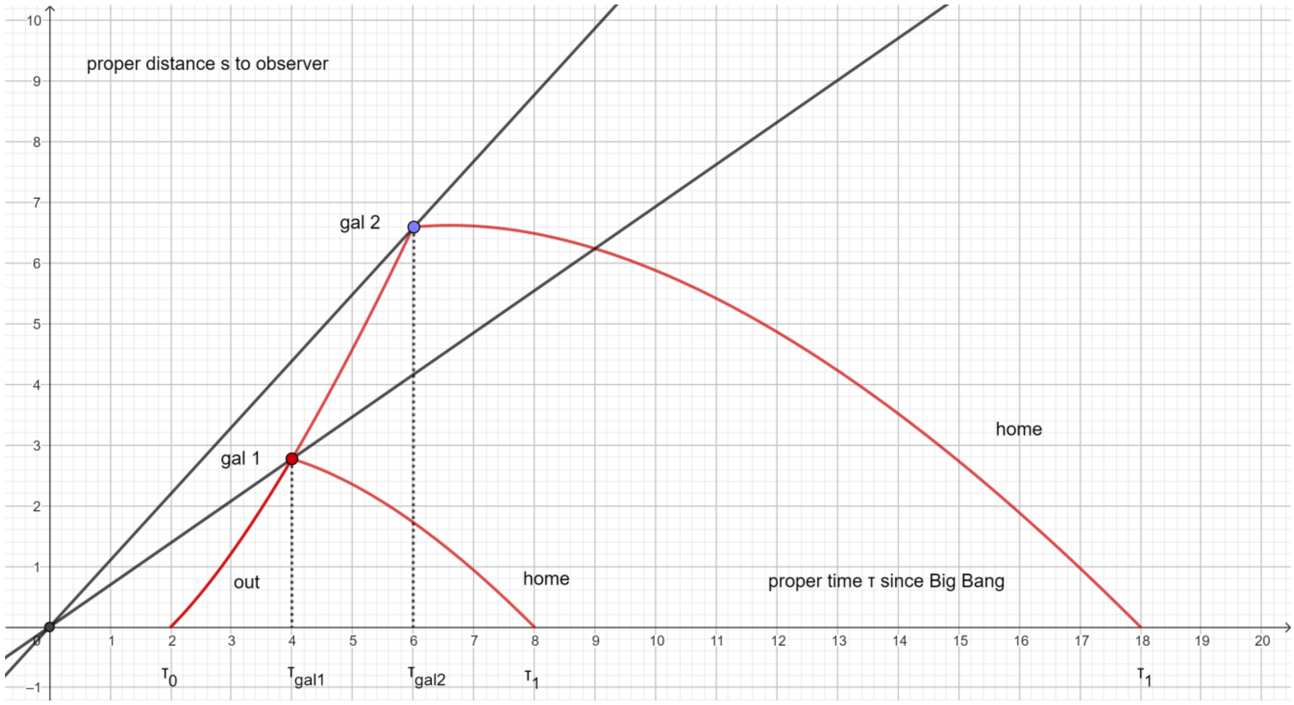

Fig. 3: GR: Tracks of Galaxies (black) og Tracks of Light (red) – the proper time τ for a given position of the galaxy is determined by a short light pulse, which is returned from the galaxy. Knowing the time of departure τ_0 and the time of return τ_1 it is possible to calculate the time τ when the light pulse hit the galaxy (formula (6) below). The proper distance s can be calculated using formula (7) below. In this way a point on the (τ, s) graph of the galaxy is determined. The constant c has been given the value of 1.

But how du we determine the connection between the three times τ_0, τ_1 and τ and the connection to the proper distance s_0 to the galaxy?

We can calculate the proper distance s_0 in two ways – one related to the journey from the observer to the galaxy – and one related to the journey back home:

(4) $$s_0 = \int_{\tau_0}^{\tau} \frac{c \cdot d\tau'}{a(\tau')} = \frac{1}{k} c \cdot ln\left(\frac{\tau}{\tau_0}\right)$$ distance calculated from the out-journey

(5) $$s_0 = \int_{\tau}^{\tau_1} \frac{c \cdot d\tau'}{a(\tau')} = \frac{1}{k} c \cdot ln\left(\frac{\tau_1}{\tau}\right)$$ distance calculated from the home-journey

Here we have related the light-distance $c \cdot d\tau'$ to the time τ_0 by dividing the light-distance with the scala-factor $a(\tau')$.

If we compare the two expressions (4) and (5) for s_0 we get the connection

$$\frac{\tau}{\tau_0} = \frac{\tau_1}{\tau}$$

And therefore:

(6) $$\tau = \sqrt{\tau_0 \cdot \tau_1}$$

As an example we take a look at fig. 3, galaxy 1. The light pulse leaves the observer at the time 2, and returns at the time 8. The light pulse hit the galaxy at the time $\tau = \sqrt{2 \cdot 8} = 4$ as shown in fig. 3.

In the figure we also see the tracks of light pulses, and we see that the velocity of the light is increasing on the way out – not surprising when the we think of the Hubble law tells us that $v_{light} = H \cdot s + c$, where the Hubble-expansion gives the contribution $H \cdot s$ to the velocity relative to the observer. Correspondingly we see that the that the light pulse on the way home 'struggles' against the expansion because

$v_{light} = H \cdot s - c$, more evident for galaxy 2.

where H is the Hubble parameter, which in the Milne model is given by the simple expression $H = 1/\tau$.

The two formulas for the velocity of light is actually differential equations because $v_{light} = s'(\tau)$. It is the solutions to these equations that are drawn I fig. 3. These solutions are given by

$$(7) \qquad s(\tau) = \pm c \cdot \tau \cdot \ln{(\tau/\tau_0)})$$

where τ_0 is where the graphs hits the time axes. The plus sign denotes solutions moving away from the observer and the minus sign denotes solutions moving the opposite way.

If we use the formulas (4) og (5) together with (6), we get

$$(8) \qquad \tau_0 = \tau \cdot e^{-\frac{k \cdot s_0}{c}}$$

$$(9) \qquad \tau_1 = \tau \cdot e^{\frac{k \cdot s_0}{c}}$$

Now it is possible to calculate the coordinates for the same event (the light pulse meets the galaxy) in our SR-system.

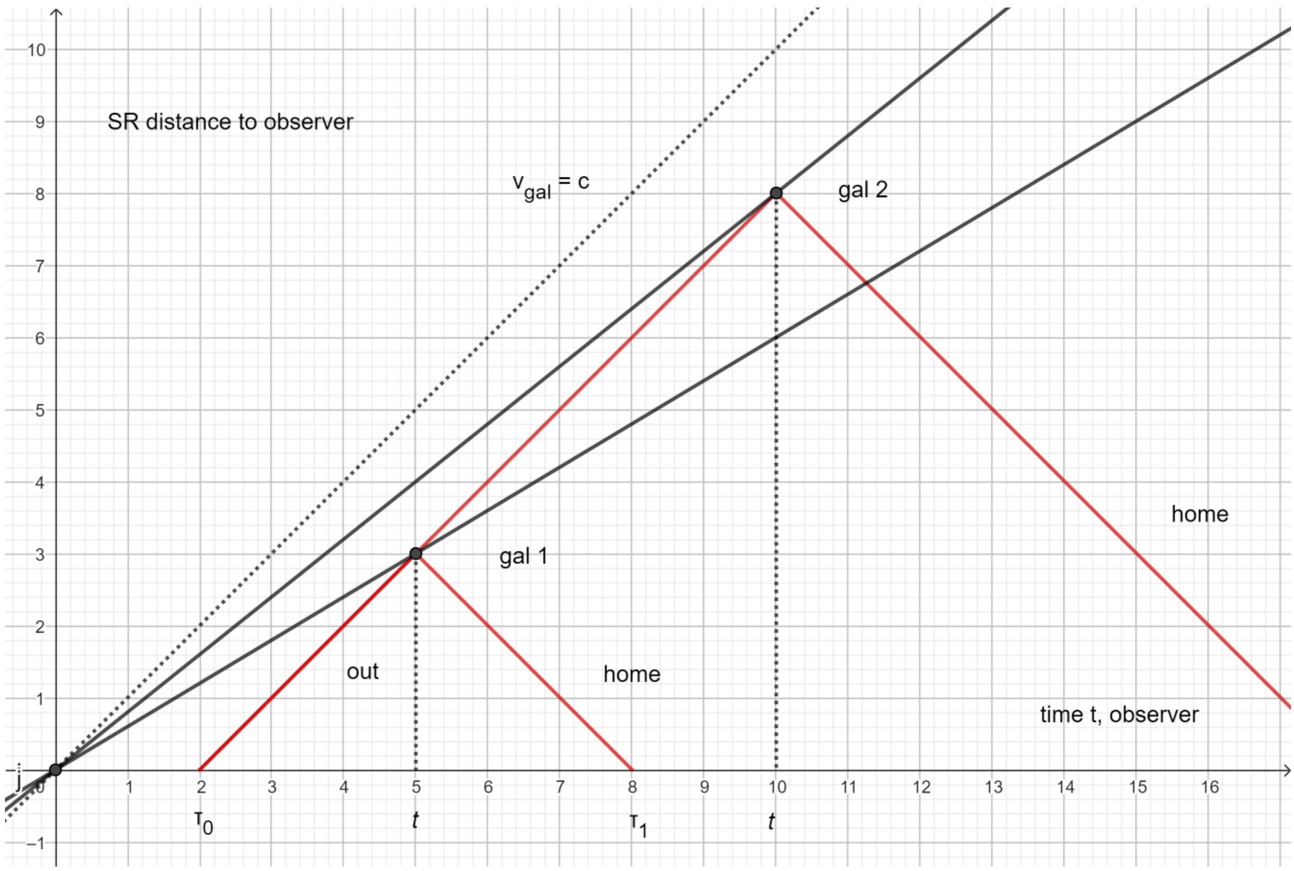

Fig. 4: SR: Tracks of galaxies and tracks of light pulses – same events as fig. 3. The time t for a galaxy position is determined by the event when the light pulse hits the galaxy and is returned to the observer. The time t where the light pulse hits the galaxy can be

calculated from the time of departure from the observer τ_0 and the time of return τ_1. See formula (11) below. The constant c is set to 1.

The two times τ_0 and $\tau_1$is measured by the same watch in the position of the observer O and can therefore be used as SR-times. The time from observer to galaxy and back again are equal, and the speed of light is c both ways. Therefore we have

(10) $\quad x = c\frac{\tau_1-\tau_0}{2}$

(11) $\quad t = \frac{\tau_1+\tau_0}{2}$

See the tacks of light in SR coordinates in fig. 4.

As an example the distance *x* for galax y 1 is $x = c\frac{\tau_1-\tau_0}{2} = 1\cdot\frac{8-2}{2} = 3,$ and the time t is $t = \frac{\tau_1+\tau_0}{2} = \frac{8+2}{2} = 5$ as shown in the figure.

Now we continue to the transformation between the coordinate systems.

We are now using the formulas (8) og (9), and we get

(12) $\quad x = c\cdot\tau\cdot\sinh\left(\frac{k\cdot s_0}{c}\right)$

(13) $\quad c\cdot t = c\cdot\tau\cdot\cosh\left(\frac{k\cdot s_0}{c}\right)$

where $\quad \sinh(x) = \frac{e^x-e^{-x}}{2}$ og $\cosh(x) = \frac{e^x+e^{-x}}{2}$.

The matematical functions $\sinh(x)$ og $\cosh(x)$ fulfills the following equation:

$$(\cosh(x))^2 - (\sinh(x))^2 = 1$$

Now using the equations (12) og (13) it follows that

(14) $\quad c^2\cdot t^2 - x^2 = c^2\cdot\tau^2$

Hereby we have expressions for both the coordinate x and the time t for the galaxy in the SR system.

The velocity of the galaxy in the SR system is (also) a constant and is given by

$$v_{SR} = \frac{x}{t} = \frac{c\cdot\tau\cdot\sinh\left(\frac{k\cdot s_0}{c}\right)}{\tau\cdot\cosh\left(\frac{k\cdot s_0}{c}\right)} = c\cdot\tanh\left(\frac{k\cdot s_0}{c}\right) = c\cdot\tanh\left(\frac{v_{GR}}{c}\right)$$

In conclusion we have

(15) $\quad v_{SR} = c\cdot\tanh\left(\frac{v_{GR}}{c}\right)$ Velocity of galaxy in GR and SR coordinates

where we have used eq. (3).

It hereby follows that as the velocity of the galaxy in GR coordinates tends to infinity the velocity of the galaxy in SR coordinates approaches c.

In fig. 5 we see the connection between v_{GR} og v_{SR}.

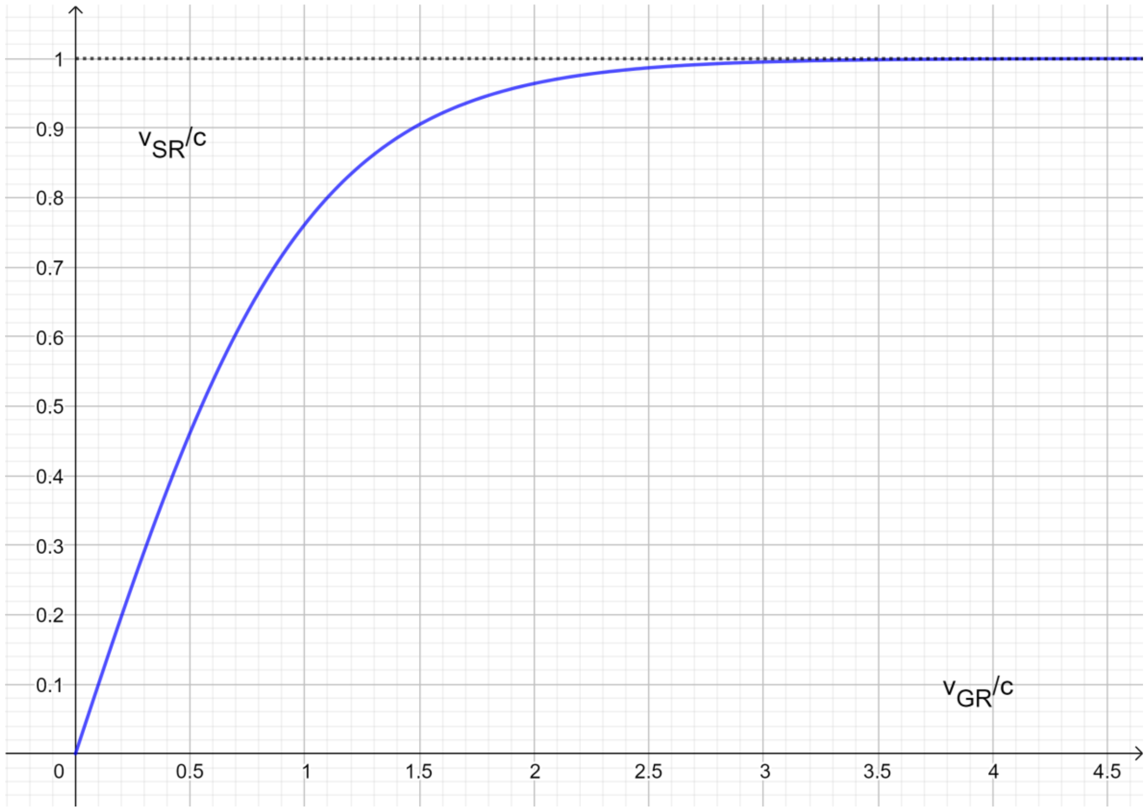

Fig. 5: v_SR vs. v_GR

As it can be seen there are no upper limit to v_{GR} but it does not mean that the galaxy moves faster than light as $v_{SR} < c$.

Redshift

Knowing the redshift *z* how do we calculate for example the time of emission, the distance to the source etc?

To find the time of emission we must use the scale factor $a(\tau)$:

(16) $$1 + z = \frac{\lambda_0}{\lambda_e} = \frac{a(\tau_0)}{a(\tau_e)}$$ redshift and scale factor

The time of emission is denoted by τ_e, the observed wavelength by λ_0 and λ_e is the wavelength as seen from the emitter galaxy (also denoted the laboratory wavelength).

In the case of the Milne model the scale factor is especially simple (eq. (2)): $a(\tau) = \tau/\tau_0$

Using this equation (16) we therefore find:

$$1 + z = \frac{\tau_0}{\tau_e}$$

The time of emission therefore is

(17) $$\tau_e = \frac{\tau_0}{1+z}$$ time of emission

From this expression we can calculate the distance from the observer to the emitter galaxy at the time of emission (eq. (7)):

(18) $$s(\tau_e) = -c \cdot \tau_e \cdot \ln\left(\frac{\tau_e}{\tau_0}\right) = c \cdot \tau_e \cdot \ln(1+z) = c \cdot \tau_0 \cdot \frac{1}{1+z} \cdot \ln(1+z)$$

where we have used equation (17).

The present distance (at the time τ_0) now can be calculated:

(19) $$s_0 = (1+z) \cdot s(\tau_e) = c \cdot \tau_0 \cdot \ln(1+z)$$

Finally we calculate the velocity of the emitter galaxy (a constant in this model):

(20) $$v = \frac{s_0}{\tau_0} = c \cdot \ln(1+z)$$ velocity of the emitter galaxy

As an example we put $z = 10$, the velocity is

(21) $$v = c \cdot \ln(1+z) = c \cdot \ln(1+10) \approx 2{,}3979 \cdot c$$ velocity at $z = 10$

But you could ask: is this faster than light?

To answer this question we change to the SR description, where we know that the speed of light is c. We apply the transformation formula (15):

(22) $$v_{SR} = c \cdot \tanh\left(\frac{v_{GR}}{c}\right) = c \cdot \tanh(2{,}3979) = 0{,}98361 \cdot c$$

It is therefore evident that the velocity of the galaxy does not exceed the speed of light.

Now that we have calculated the velocity of the emitter galaxy in SR coordinates, we can of course also calculate the redshift (which we already know!) by using the wellknown SR formula:

(23) $$1 + z = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} = \sqrt{\frac{1+0{,}98361}{1-0{,}98361}} = 11{,}00$$ redshift in SR

The calculation could of course have been done exact by not using the approximation in (21).

This shows that the description can be done in both coordinate systems as we can translate from GR coordinates/velocities to SR coordinates/velocities and vice versa.

So before you proclaim *faster-than-light velocity* in GR you have got to ask yourself: could it be the coordinate system being used here that have given the strange result? Or from ref. 2:

'Next time you hear of something strange going on in cosmology remember to think 'Is this just because of the choice of coordinate system?'

Redshift: Doppler shift upon Doppler shift upon…

As a final point we take a look at light from distant sources exhibiting large values of redshift, also called cosmological redshift, and we will see how it is possible to describe arbitrarily large redshifts as a long series of smaller Doppler shifts (or to be more precise: the large value of the redshift happens as a continuous Doppler shift in the expanding Universe)

We here take a look at a series of redshifts happening to the light on its long journey in the Universe from emitter to the observer.

The wavelenght of light as seen from the emitter galaxy we denote by λ_e and the wavelength as measured by the observer we denote by λ_{obs}.

The wavelenght as seen from the first galaxy/observer passed by the light after it have left the emitter galaxy we denote by λ_1, and as seen by the second galaxy/observer we denote the wavelength by λ_2 etc.

And the wavelenght as seen by the last galaxy/observer before the arrival at our telescope we will denote by λ_{n-1}.

The light will all in all have undergone *n* redshifts when it arrives at our telescopes.

We will use the identity

(24) $$\frac{\lambda_{obs}}{\lambda_e} = \frac{\lambda_{obs}}{\lambda_{n-1}} \cdot \frac{\lambda_{n-1}}{\lambda_{n-2}} \cdot \frac{\lambda_{n-2}}{\lambda_{n-3}} \cdot \ldots \cdot \frac{\lambda_2}{\lambda_1} \cdot \frac{\lambda_1}{\lambda_e}$$

In this equation we introduce the global redshift z, together with the local redshifts Δz_i:

(25) $$1 + z = (1 + \Delta z_n) \cdot (1 + \Delta z_{n-1}) \cdot (1 + \Delta z_{n-2}) \cdot \ldots \cdot (1 + \Delta z_2) \cdot (1 + \Delta z_1)$$

If we assume that all the local redshifts are equal (and we denote this value by Δz) we find

(26) $$1 + z = (1 + \Delta z)^n$$ global redshift z and local redshift Δz

which gives us the following:

(27) $$\ln (1 + z) = n \cdot \ln (1 + \Delta z)$$

If the number *n* is big then the value of Δz will be small, and we can make the approximation $\ln (1 + \Delta z) \approx \Delta z$.

Therefore:

$$\ln (1 + z) \approx n \cdot \Delta z$$

or

(28) $$\Delta z \approx \frac{\ln (1+z)}{n}$$ local redshift

If we as an example divide the global redshift in 10 000 parts it follows

(29) $$\Delta z \approx \frac{\ln(1+10)}{10000} = 0{,}00023979$$

This value of the redshift corresponds after the Doppler law to the velocity

(30) $$v = \Delta z \cdot c = 0{,}00023979 \cdot 300\ 000\ \mathrm{km/s} = 71{,}937\ \mathrm{km/s}$$

which is a totally nonrelativistic velocity.

If the galaxies do not change their velocity over time (the Milne model) then the emitter galaxy relative to our telescopes will have the velocity

(31) $$v_e = n \cdot v = 10\ 000 \cdot 71{,}937\ \mathrm{km/s} = 719\ 370\ \mathrm{km/s} = 2{,}3979\ c$$

Note that this is the same velocity we found in the Milne model above.

In the Milne model (and only in this) we can translate this velocity to special relativity SR by using

$$(32) \qquad v_{e,\ SR} = c \cdot \tanh(2{,}3979) = 0{,}98361\ c$$

That is – not faster than the speed of light! (as we already have concluded above)

The reason for this apparent faster-than-light velocity in GR coordinates is as already mentioned the distance concept: the distance from the observer to a galaxy is the sum of locally measured proper distances. The distances are not – in spite of the large velocities involved – Lorentz shortened as in the special theory of relativity. Therefore there is no upper limit to the velocity.

Could we have calculated this SR velocity solely from the knowledge that the relative velocity of the galaxies pairwise is given by (30)?

The answer is yes – as we will see below.

Here we are going to 'add' velocities in SR, and this is done by using the Lorentz transformation. We will add 10 000 velocities each of size 71,937 km/s.

This totally nonrelativistic velocity valid for example for the nearest galaxy nearest to our observer can be used in both the GR system and the SR system as we have the transformation formula (15). For nonrelativistic velocities there are (almost) no difference in the two systems.

With the intension to add velocities in SR we will have to use the formula

$$(33) \qquad v_{n,SR} = \frac{\left(1+\frac{v}{c}\right)^n - \left(1-\frac{v}{c}\right)^n}{\left(1+\frac{v}{c}\right)^n + \left(1-\frac{v}{c}\right)^n} \cdot c \qquad \text{n repeated Lorentz-transformations}$$

Where the second galaxy are moving away from the first by the velocity *v*, and the third galaxy are moving away from the second by the velocity *v* etc. We have by using formula (33) added *n* identical velocities by using the principles of the Lorentz transformation for each step.

The formula can be proven by using the concept of *rapidity* (ref. 3). Or it can be proven by using the principle of induction!

To do this we have to use the formula from SR where we add two parallel velocities:

$$v_{SR} = \frac{u+v}{1+\frac{u \cdot v}{c^2}} \qquad \text{addition of the velocity u to the velocity v in SR}$$

The velocity v is measured in the inertial system S_1, and the velocity u is measured in the inertial system S_2 that is moving by the velocity v as seen from the system S_1.

v_{SR} is the resulting velocity as seen from the system S_1. So good luck with the principle of induction!

Now we use (33) with the value $\frac{v}{c} = \frac{71{,}937 \text{ km/s}}{300\ 000 \text{ km/s}} = 0{,}00023979$ and we have

$$(34) \qquad v_{10000,SR} = \frac{(1+0{,}00023979)^{10000} - (1-0{,}00023979)^{10000}}{(1+0{,}00023979)^{10000} + (1-0{,}00023979)^{10000}} \cdot c = 0{,}98361\ c$$

- That is, the same SR velocity as in (32).

This by no way a surprise!

To argue for this we remind the reader of the lemma $(1 + x/n)^n \rightarrow e^x$ as $n \rightarrow \infty$

Put here $x = \ln(11)$ and show that the limit above is $120/122$.

Hereby we see the essential difference in the way to add velocities in Milne GR coordinates and in SR coordinates.

The difference is reflected in the formulas (31) og (34).

This limiting value of the velocity corresponds to the redshift 10 as we started out assuming.

The SR formula for the redshift (35) below can of course be transformed to express the velocity by the redshift:

$$(35) \qquad 1 + z = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$ SR-formula for the redshift

$\frac{v}{c}$ is isolated:

$$(36) \qquad \frac{v}{c} = \frac{(1+z)^2-1}{(1+z)^2+1}$$

We see from this formula that no matter the value of the redshift the velocity will never exceed the value *c*.

Putting $z = 10$ in (36) gives:

$$(37) \qquad \frac{v}{c} = \frac{(1+10)^2-1}{(1+10)^2+1} = \frac{120}{122} = 0{,}98361 \ldots$$

- as expected

When there are gravitational forces involved the velocities of the galaxies are of course not constant in time not even in the Hubble flow. Therefore we cannot carry through the calculation of the velocity of the emitter galaxy as we did above. Never the less the light from a remote galaxy will undergo a continuous Doppler shift on its way to the next galaxy in the Hubble flow – until the light finally hits our telescopes.

Ref. 1: SIMON OLLING REBSDORF - Milnes kosmofysik ISSN: 1600-7433 Aarhus Universitet

Ref. 2: Cosmology, Special Relativity and the Milne Universe (chronon.org)

Ref. 3: https://phys.libretexts.org/Bookshelves/University_Physics/Mechanics_and_Relativity_(Idema)/11%3A_Lorentz_Transformations/11.04%3A_Rapidity_and_Repeated_Lorentz_Transformations

Ref. 4: http://universeinproblems.com/index.php/The_Milne_Universe

Ref. 5: https://www.bao.am/seminars/pdf/2022/17012022.pdf